# HOMOTOPY CLASSES OF MAPS BETWEEN KNASTER CONTINUA

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Abstract. By a Knaster continuum we understand the inverse limit of copies of $[0, 1]$ with open bonding maps. We prove that for any two Knaster continua $K_1$ and $K_2$, there are $2^{\aleph_0}$ distinct homotopy types of maps of $K_1$ onto $K_2$ that map the endpoint of $K_1$ to the endpoint of $K_2$.

## 1. Introduction

Let $\mathbb{R}$ denote the set of real numbers and let $I$ denote the interval $[0, 1]$. For any real number $t$, let $[t]$ denote the greatest integer less than or equal to $t$. Let $v : \mathbb{R} \to I$ be defined by the formula

$$v\,(t) = \begin{cases} t - [t]\,, & \text{if } [t] \text{ is even} \\ [t] + 1 - t, & \text{if } [t] \text{ is odd.} \end{cases}$$

For each positive integer $n$, let $g_n : I \to I$ be defined by the formula $g_n\,(t) = v\,(nt)$. Observe that $g_n$ stretches $n$ times and then folds the resulting interval $[0, n]$ onto $[0, 1]$. The map $g_2$ is the very well known "roof-top" map. For any two positive integers $m$ and $n$, $g_m \circ g_n = g_{mn}$. Consequently, $g_n$ and $g_m$ commute (see, for example, [8, Proposition 2.2]).

Let $N = \{n_1, n_2, \dots\}$ be a sequence of integers $> 1$. Consider the inverse sequence

$$(*) \qquad I \xleftarrow{g_{n_1}} I \xleftarrow{g_{n_2}} I \xleftarrow{g_{n_3}} I \xleftarrow{g_{n_4}} \dots$$

By the Knaster continuum associated with the sequence $N$ we will understand the inverse limit of (*). Observe that the same Knaster continuum can be associated with two different sequences. For example the inverse limit does not change if we replace $N$ by $\{n_1 n_2 \dots n_{j_1}, n_{j_1+1} \dots n_{j_2}, \dots\}$. However, it should be noted here that there are $2^{\aleph_0}$ topologically distinct Knaster continua [2](see also [14]).

For a Knaster continuum $K$, let $e$ denote the endpoint $(0, 0, \dots)$. By $\pi_i$ we will understand the projection of $K$ onto the $i$-th component in the inverse system $(i = 0, 1, \dots)$.

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Let $S^1$ denote the unit circle in the complex plane and let $\tilde{g}_n : S^1 \to S^1$ be defined by $\tilde{g}_n(z) = z^n$. By the solenoid associated with the sequence $N$ we will understand the inverse limit of $S^1 \xleftarrow{\tilde{g}_{n_1}} S^1 \xleftarrow{\tilde{g}_{n_2}} S^1 \xleftarrow{\tilde{g}_{n_3}} S^1 \xleftarrow{\tilde{g}_{n_4}} \dots$. We will use $e$ to denote $(1, 1, \dots)$.

Note that the same letter $e$ is used here to denote different objects: $(0, 0, \dots)$ in different Knaster continua and $(1, 1, \dots)$ in different solenoids. For instance, if $f$ is a map between two Knaster continua $K_1$ and $K_2$, then the equality $f(e) = e$ means that $f$ maps the endpoint $(0, 0, \dots)$ of $K_1$ to the endpoint $(0, 0, \dots)$ of $K_2$.

Knaster continua and solenoids are the simplest examples of indecomposable continua. They appear naturally as attractors of dynamical systems (see for example [1] and [7]). The goal of this paper is to study homotopy classes of maps between Knaster continua. It follows from [12] (see also [4] and [10]) that each map between two Knaster continua is homotopic to a map induced from a commutative diagram. Hopotopy classes of maps between Knaster continua coincide, therefore, with those of the induced map. The easy examples of commutative diagrams could be constructed by using the commutativity of $g_n$ and $g_m$. We will first consider such examples.

Let $M = \{m_1, m_2, \dots\}$ be another sequence of positive integers. Suppose $\{k_0, k_1, k_2, \dots\}$ is the third sequence of positive integers such that $k_{i-1} n_i = m_i k_i$ for each positive integer $i$. Then the following diagram commutes.

(**)
$$\begin{tikzcd}
I \arrow[d, "g_{k_0}"'] & I \arrow[l, "g_{n_1}"'] \arrow[d, "g_{k_1}"'] & I \arrow[l, "g_{n_2}"'] \arrow[d, "g_{k_2}"'] & I \arrow[l, "g_{n_3}"'] \arrow[d, "g_{k_3}"'] & \dots \arrow[l, "g_{n_4}"'] \\
I & I \arrow[l, "g_{m_1}"] & I \arrow[l, "g_{m_2}"] & I \arrow[l, "g_{m_3}"] & \dots \arrow[l, "g_{m_4}"]
\end{tikzcd}$$

Let $K_1$ and $K_2$ be Knaster continua associated with the sequences $N$ and $M$, respectively. Observe that the diagram (**) induces a continuous map $f : K_1 \to K_2$. Note that if $x = (x_0, x_1, x_2, \dots)$ then $f(x) = (g_{k_0}(x_0), g_{k_1}(x_1), g_{k_2}(x_2), \dots)$.

We will say that a map $g : K_1 \to K_2$ is *naturally induced* if there are sequences $\{j_0, j_1, j_2, \dots\}$ and $\{i_0, i_1, i_2, \dots\}$ such that $0 \le j_0 < j_1 < j_2 < \dots$ and

$$g(x) = (g_{i_0}(x_{j_0}), g_{i_1}(x_{j_1}), g_{i_2}(x_{j_2}), \dots).$$

The map induced by the diagram (**) is an example of a naturally induced map. In the general case, the vertical arrows in the diagram (**) may be replaced by diagonal ones. The following proposition is a simple consequence of the definition.

**Proposition 1.1.** *Suppose $K_1$ and $K_2$ are Knaster continua associated with the sequences $\{n_1, n_2, \dots\}$ and $\{m_1, m_2, \dots\}$, respectively. Let $\{i_0, i_1, \dots\}$ and $\{j_0, j_1, \dots\}$ be two sequences of non negative integers with $j_0 < j_1 < j_2 < \dots$. Let $g : K_1 \to K_2$ be the naturally induced map defined by*

$$g(x) = (g_{i_0}(x_{j_0}), g_{i_1}(x_{j_1}), g_{i_2}(x_{j_2}), \dots)$$

*for each $x = (x_0, x_1, x_2, \dots) \in K_1$. Then*

$$i_k = \frac{i_0 n_{j_0+1} n_{j_0+2} \dots n_{j_k}}{m_1 m_2 \dots m_k}$$

*for each positive integer $k$.*

**Corollary 1.2.** *Suppose $K_1$ and $K_2$ are two Knaster continua and $\pi_0''$ denote the projection of $K_2$ onto the $0-th$ factor in the inverse sequence defining $K_2$. Let $g$ and $f$ be two naturally induced maps between of $K_1$ into $K_2$ such that $\pi_0'' \circ g = \pi_0'' \circ f$. Then $g = f$.*

**Corollary 1.3.** *The set of naturally induced maps between two Knaster continua is countable.*

The notion of naturally induced maps may be introduced for solenoids. Suppose $\Sigma_1$ and $\Sigma_2$ are solenoids associated with the sequences $N$ and $M$, respectively. We will say that a map $\tilde{g} : \Sigma_1 \to \Sigma_2$ is *naturally induced* if there are sequences $\{j_0, j_1, j_2, \dots\}$ and $\{i_0, i_1, i_2, \dots\}$ such that $0 \leq j_0 < j_1 < j_2 < \dots$ and

$$\tilde{g}(x) = (\tilde{g}_{i_0}(x_{j_0}), \tilde{g}_{i_1}(x_{j_1}), \tilde{g}_{i_2}(x_{j_2}), \dots).$$

The statements corresponding to 1.1, 1.2 and 1.3 are true for solenoids. Observe that any naturally induced map, either between two Knaster continua or between two solenoids, maps $e$ to $e$. Suppose $f$ is a map with the property that $f(e) = e$. Must $f$ be homotopic to a naturally induced map? In case of solenoids, the answer is positive. The following proposition follows from [13], [5] and [6] (see also [9, Proposition 3]).

**Proposition 1.4.** *Suppose $f$ is a map of a solenoid $\Sigma_1$ into a solenoid $\Sigma_2$ such that that $f(e) = e$. Then $f$ is homotopic to a naturally induced map.*

A similar statement for Knaster continua would be false. For instance, any Knaster continuum $K_1$ can be always mapped onto any Knaster continuum $K_2$ (see [11]). On the other hand, there is no naturally induced maps of $K_1$ of $K_2$ if the sequences defining $K_1$ and $K_2$ have different prime factors (see Proposition 1.1).

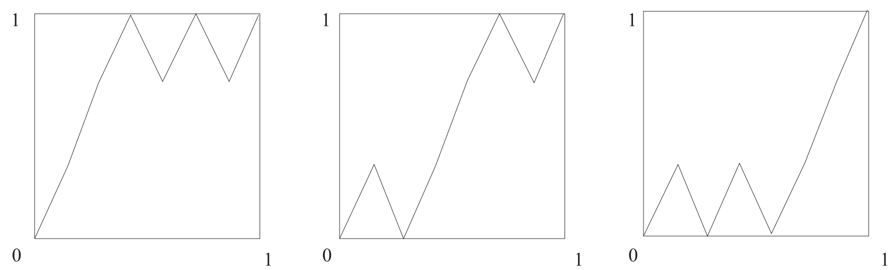
It follows from Proposition 1.4 that there is only countably many homotopy classes of maps between solenoids mapping $e$ into $e$. One could expect the corresponding theorem for Knaster continua since the mapping structure of solenoids is usually richer. After all, any solenoid can be mapped onto any Knaster continuum, but there is no continuous map of a chainable continuum onto a solenoid. We will show, however, that in case of Knaster continua there are uncountably many homotopy types of maps mapping the endpoint of the domain onto the endpoint of the range. More precisely, we will prove the following theorem.

**Theorem 1.5.** *For any Knaster continua $K_1$ and $K_2$, there are $2^{\aleph_0}$ distinct homotopy types of maps $f : K_1 \to K_2$ such that $f(e) = e$.*

It should be noted here that the maps in the above theorem cannot be replaced by homeomorphisms even in the case where $K_1 = K_2$. It follows from [3, Theorem 3.3, p. 53] that there is only countably many homotopy types of homeomorphisms of any Knaster continuum onto itself.

## 2. Constructing Maps between Knaster continua.

We will prove Theorem 1.5 by defining $2^{\aleph_0}$ maps of $K_1$ onto $K_2$ induced by commutative diagrams. We will construct the vertical arrows $f_0, f_1, f_2, \dots$ one by one. We will start from $f_0$ equal to the identity on $I$. We will then use Lemma 2.1 repeatedly to define $f_1, f_2, \dots$ in many different ways. The figure depicts the graphs of three maps $f_1$ such that $f_0 \circ g_7 = g_3 \circ f_1$. Similar maps will be used in Lemma 2.1.




Figure

**Lemma 2.1.** *Let $m$, $n$ and $q$ be positive integers such that $(m+2)\,q \le n$. Suppose $f_0$ is a map of $I$ onto itself. Then for each $i = 0, 1, \dots, q-1$ there is a map $f_1 : I \to I$ such that*

(1) $f_1(0) = 0$ *if* $f_0(0) = 0$,
(2) $f_0 \circ g_n = g_m \circ f_1$,
(3) $f_1\left(\left[0, \frac{i}{q}\right]\right) \subset \left[0, \frac{1}{m}\right]$,
(4) $f_1\left(\left[\frac{i}{q}, \frac{i+1}{q}\right]\right) = I$ *and*
(5) $f_1\left(\left[\frac{i+1}{q}, 1\right]\right) \subset \left[\frac{m-1}{m}, 1\right]$.

*Proof of* 2.1. Let $k$ be the least non negative integer such that $\frac{k}{n} \ge \frac{i}{q}$. Since $(m+2)\,q \le n$, it follows that $\frac{k+m+1}{n} < \frac{i+1}{q}$ and consequently

$$\left[\frac{k}{n}, \frac{k+m+1}{n}\right] \subset \left[\frac{i}{q}, \frac{i+1}{q}\right].$$

Let $a, b \in I$ be such that $f_0(a) = 0$ and $f_0(b) = 1$. Since $g_n$ maps each of the intervals $\left[\frac{k+j}{n}, \frac{k+j+1}{n}\right]$ onto $I$, for each $j = 0, \dots, m$, there is a point $t_j \in \left[\frac{k+j}{n}, \frac{k+j+1}{n}\right]$ such that $g_n(t_j) = a$ if $j$ is even and $g_n(t_j) = b$ if $j$ is odd. Define $f_1$ by

$$f_1(t) = \frac{1}{m}\begin{cases} f_0 \circ g_n(t), & \text{for } 0 \le t \le t_1 \\ j+1 - f_0 \circ g_n(t), & \text{for an odd } j = 1, \dots, m-1 \text{ and } t_j \le t \le t_{j+1} \\ j + f_0 \circ g_n(t), & \text{for an even } j = 2, \dots, m-1 \text{ and } t_j \le t \le t_{j+1} \\ m-1+f_0 \circ g_n(t), & \text{for } t_m \le t \le 1 \text{ if } m \text{ is odd} \\ m - f_0 \circ g_n(t), & \text{for } t_m \le t \le 1 \text{ if } m \text{ is even.} \end{cases}$$

One can verify that so defined $f_1$ has the required properties. (Use the equality $g_m(x) = v(mx)$ to verify 2.)

**Construction of $f_t$.** Let $K_1$ and $K_2$ be the Knaster continua associated with sequences $\{n_1, n_2, \dots\}$ and $\{m_1, m_2, \dots\}$, respectively. Observe that $K_1$ does not change if the sequence $\{n_1, n_2, \dots\}$ is replaced by a sequence of products of finite blocks of consecutive elements of its elements. So we may assume that

$$(m_j + 2)\,j < n_j \quad \text{for each positive integer } j.$$

Let $t \in I$. For each positive integer $j$, let $i[t,j]$ be an integer such that $0 \le i[t,j] < j$ and $t \in \left[\frac{i[t,j]}{j}, \frac{i[t,j]+1}{j}\right]$. We will define a sequence $f_0^t, f_1^t, f_2^t, \ldots$ of maps of $I$ onto itself such that

(1) $f_j^t(0) = 0$ ,
(2) $f_{j-1}^t \circ g_{n_j} = g_{m_j} \circ f_j^t$,
(3) $f_j^t\left(\left[0, \frac{i[t,j]}{j}\right]\right) \subset \left[0, \frac{1}{m_j}\right]$,
(4) $f_j^t\left(\left[\frac{i[t,j]}{j}, \frac{i[t,j]+1}{j}\right]\right) = I$ and
(5) $f_j^t\left(\left[\frac{i[t,j]+1}{j}, 1\right]\right) \subset \left[\frac{m_j-1}{m_j}, 1\right]$.

for each positive integer $j$.

Let $f_0^t$ be the identity on $I$. Suppose $f_0^t, \ldots, f_{j-1}^t$ have been defined. Use Lemma 2.1 with $m = m_j$, $n = n_j$, $q = j$, $f_0 = f_{j-1}^t$ and $i = i[t,j]$. Set $f_j^t$ to be $f_1$ obtained from the lemma. Observe that conditions 1-5 follow from the corresponding conditions in the lemma.

Let $f^t : K_1 \to K_2$ be the function induced by the sequence $f_0^t, f_1^t, f_2^t, \ldots$.

The following proposition is a simple consequence of the construction.

**Proposition 2.2.** *For each $t \in I$, $f^t$ is a continuous map of $K_1$ onto $K_2$ such that $f^t(e) = e$.*

**Proposition 2.3.** *Suppose $t, s \in I$ and $t \neq s$. Then $f^t$ is not homotopic to $f^s$.*

*Proof of* 2.3. Suppose $f^t$ is homotopic to $f^s$ for some $t, s \in I$ such that $t < s$. Let $H : K_1 \times I \to K_2$ be the homotopy between $f^t$ and $f^s$. Consider the homotopy $h = \pi_0'' \circ H : K_1 \times I \to I$, where $\pi_0''$ denote the projection of $K_2$ onto the $0-th$ factor in the inverse sequence defining $K_2$. By compactness, there is a sequence of numbers $z_0 = 0 < z_1 < \cdots < z_{\ell-1} < z_\ell = 1$ such that the set $h(\{x\} \times [z_{k-1}, z_k])$ does not contain $I$ for each $x \in K_1$ and $k = 1, \ldots, \ell$.

Let $j$ be an integer such that

$$m_1 m_2 \ldots m_{j-1} > \ell \quad \text{and} \quad \frac{3}{j} < s - t.$$

Since $2j < n_j$, there is an integer $q$ such that

$$\frac{i[t,j]+1}{j} \le \frac{2q}{n_j} \le \frac{i[t,j]+2}{j}.$$

By condition 5 of the construction of $f^t$, we have that

$$\text{(i)} \qquad f_j^t\left(\frac{2q}{n_j}\right) \in \left[\frac{m_j-1}{m_j}, 1\right].$$

Since $\frac{3}{j} < s - t$ and $t \in \left[\frac{i[t,j]}{j}, \frac{i[t,j]+1}{j}\right]$, we have that $\frac{i[t,j]+3}{j} < s$. Since $s \in \left[\frac{i[s,j]}{j}, \frac{i[s,j]+1}{j}\right]$, it follows that $\frac{i[t,j]+2}{j} < \frac{i[s,j]}{j}$ and consequently $\frac{2q}{n_j} \in \left[0, \frac{i[s,j]}{j}\right]$. By condition 3 of the construction of $f^s$, we have that

$$\text{(ii)} \qquad f_j^s\left(\frac{2q}{n_j}\right) \in \left[0, \frac{1}{m_j}\right].$$

Since $g_{n_j}\left(\frac{2q}{n_j}\right)=0$ and $f^s_{j-1}(0)=0$, condition 2 of the construction of $f^s$ implies that $g_{m_j}\circ f^s_j\left(\frac{2q}{n_j}\right)=0$. It follows from (ii) that

$$f^s_j\left(\frac{2q}{n_j}\right)=0. \tag{iii}$$

Let $\pi'_j$ denote the projection of $K_1$ onto the $j-th$ factor in the inverse sequence defining $K_1$. Similarly, let $\pi''_j$ denote the projection of $K_2$ onto the $j-th$ factor in the inverse sequence defining $K_2$.

Take a point $y\in K_1$ such that $\pi'_j(y)=\frac{2q}{n_j}$. It follows from (i) and (iii) that

$$\pi''_j\circ f^s(y)=0 \quad \text{and} \quad \pi''_j\circ f^t(y)\in\left[\frac{m_j-1}{m_j},1\right]. \tag{iv}$$

Let $p$ denote the product $m_1m_2\dots m_{j-1}$ and let $r=pm_j$. By (iv), there are numbers $w_0,w_1,\dots w_p\in I$ such that $0=w_0<w_1<\dots<w_p$ and

$$\pi''_j\circ H(y,w_\lambda)=\frac{\lambda}{r}\quad\text{for each } \lambda=0,1,\dots,p.$$

Observe that $h(y,w_\lambda)=\pi''_0\circ H(y,w_\lambda)=g_r\circ\pi''_0\circ H(y,w_\lambda)=g_r\left(\frac{\lambda}{r}\right)$. Thus,

$$\{h(y,w_{\lambda-1}),h(y,w_\lambda)\}=\{0,1\}\quad\text{for each } \lambda=1,\dots,p. \tag{v}$$

Recall that $p>\ell$ by the choice of $j$. Hence, there are integers $k$ and $\lambda$ such that $1\le k\le\ell$, $1\le\lambda\le p$ and

$$w_{\lambda-1},w_\lambda\in[z_{k-1},z_k].$$

Now, (v) contradicts the choice of $z_0,z_1,\dots,z_\ell$.

*Proof of Theorem* 1.5. The theorem follows from Propositions 2.2 and 2.3.

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